# Real-time analysis of the telegrapher's equation for tunneling processes

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Based on a close analogy with an *RLC* circuit, a model for interpreting delay times in forbidden regimes (tunneling) is formulated, avoiding the analytical continuation into imaginary time. In this way, a reasonable description of experimental data, which were previously reported for a waveguide propagation below the cutoff frequency at ~9.5 GHz, is obtained.

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In a previous paper [1], a stochastic modeling of tunneling processes, based on the Kac's solution of the telegrapher's equation [2] suitably continued in an analytical manner into imaginary time [3], has been improved. The model described in Ref. [1] demonstrated its ability to interpret results of delay time in microwave experiments performed in both allowed and forbidden spectral regions, namely, above and below the cutoff frequency of narrow waveguide sections of different lengths.

The purpose of the present work is to further investigate this problem in an attempt at finding a different formulation of the model, avoiding the analytical continuation into imaginary time; that is, by considering the time in its natural characteristic as a real variable.

At first, we want to show that the telegrapher's equation

$$\frac{\partial^2 F}{\partial t^2} + 2a\frac{\partial F}{\partial t} - v^2\frac{\partial^2 F}{\partial x^2} = 0, \tag{1}$$

where $F$ represents the voltage for a real line or, more generally, a field component, can be solved in a relatively simpler way (with respect to that of Ref. [2]), by using the method of the Laplace transforms. Denoting the transform of $F(x,t)$ by $f(x,s)$, the Laplace transform of the wave equation (1) is [4,5]

$$(s^2 + 2as)f(x,s) - (s+2a)\Phi(x,0) - \left(\frac{d\Phi}{dt}\right)_{t=0} - v^2\frac{d^2 f}{dx^2} = 0, \tag{2}$$

where $\Phi(x,t)$ is a solution of Eq. (1) without dissipation ($a=0$). For a simple progressive wave of the type $\Phi(x,t)=\sin(x-vt)$ [6], Eq. (2) becomes

$$\frac{d^2 f}{dx^2} - \frac{1}{v^2}(s^2+2as)f + \frac{s+2a}{v^2}\sin x - \frac{1}{v}\cos x = 0. \tag{3}$$

A solution of Eq. (3) can be found in the form $f=A\sin x + B\cos x$, with $A$ and $B$ functions of $s$. Furthermore, by using a standard procedure, we easily obtain [7]

$$f(x,s) = \frac{(s+2a)\sin x - v\cos x}{v^2 + s^2 + 2as}. \tag{4}$$

The inverse Laplace transform of Eq. (4) is the solution of Eq. (1) sought, namely, [4]

$$F(x,t) = e^{-at}\left[\sin x\cos(w_1 t) + \frac{a}{w_1}\sin x\sin(w_1 t) - \frac{v}{w_1}\cos x\sin(w_1 t)\right], \tag{5}$$

where $w_1=\sqrt{v^2-a^2}$, with $v \geqslant a$. When $v \leqslant a$, the circular functions are replaced by hyperbolic functions, and the argument $w_1 t$ becomes $w_2 t$ with $w_2=\sqrt{a^2-v^2}$ (for $v^2>0$), or $w_3 t$ with $w_3=\sqrt{a^2+|v^2|}$ (for $v^2<0$).

The result expressed by Eq. (5) is identical to the one obtained by using the method of Ref. [2] [see Eq. (13) in Ref. [5], for $\alpha=1$ and $\beta=0$, or Eq. (7) in Ref. [3]]. However, although original and elegant, this method implies many calculations, and it is undoubtedly more complicated than the one adopted here, which makes it possible to find a solution of Eq. (1) immediately.

Equation (5) and related expressions (for $v<a$ and $v^2<0$) were at the basis of the model formulated in Ref. [3]. There, however, in order to avoid aperiodic functions of time, which were considered not suitable for interpreting delay time results (because of the difficulty in defining a shift between two functions of this kind), we adopted the analytical continuation into imaginary time ($t\to i\tau$) in order to recover pseudo progressive waves of the $\sin(x-w_{2,3}\tau)$ type [8]. As anticipated before, in this paper we will try to use the solution of the telegrapher's equation [Eq. (5)] by maintaining the reality of the time, and avoiding the analytical continuation into the imaginary one.

The shape of $F(x,t)$ is shown in Fig. 1(a) for the case of $v>a$ and in Fig. 1(b) for $v<a$. In the first case, the delay in going from $x=0$ to $x=1$ is clearly identified with the dephasing between the two wave forms. In the second case, this identification is less evident. A rough criterion for evaluating the delay, when $x$ increases, can be obtained by considering the intersection of the tangent at the initial portion of the curve with the time axis [dotted lines in Fig. 1(b)], which increases with an increase in the spatial coordinate $x$. This criterion is rather naive; however, it would be exact only if the curves could be identified with single exponential functions, the time constants of which are given by the above intersection.

A more refined criterion can be established on the basis of an analogy of the system under study (a narrowed waveguide section which is the homolog of a rectangular potential bar-

 

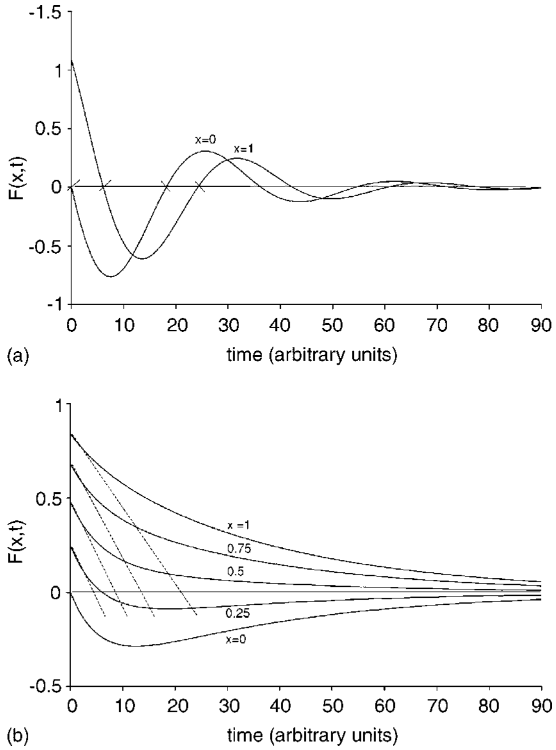


FIG. 1. Shapes of the function $F(x,t)$ as given by Eq. (5): (a) for the case of $v=0.2$ and $a=0.05$, where the delay, from $x=0$ to $x=1$, is clearly evidenced; (b) for the case of $v=0.07$ and $a=0.1$, where the delay for the different $x$ values is roughly obtained by the intersection of the tangent by the time axis.

rier) with an electric circuit of an *RLC* type, see Fig. 2. In fact, a section of waveguide can be considered as a section of a line which, in turn, is equivalent to an electric network constituted by resonant circuits of different resonant frequencies. If we limit ourselves to the neighborhood of one resonant frequency, it appears natural to consider a single circuit [7]. According to the analysis of Ref. [9], the response $e_o(t)$ to a step signal of amplitude $E$ can be approximated by the sum of two exponential functions of the type

$$e_o(t) = E[\exp(p_1 t) - \exp(p_2 t)], \tag{6}$$

where $p_{1,2}$ are the roots of the characteristic equation given by [Eq. (2.34) in Ref. [9]]

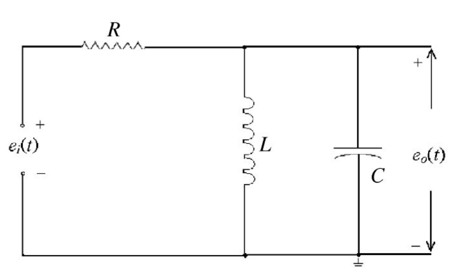


FIG. 2. An *RLC* circuit which is the analogous of the system under study (from Ref. [9]).

$$p_{1,2} = -\frac{1}{2RC} \pm \left[\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}\right]^{1/2}. \tag{7}$$

By putting

$$K = \left(\frac{1}{2R}\right)\sqrt{\frac{L}{C}},$$

the shape of Eq. (6) for overdamped cases $K>1$ (see Figs. 2.24 and 2.30 in Ref. [9]) is indeed the same type as that in Fig. 1(b). In underdamped cases, for $K<1$, the response is a damped oscillation (see Figs. 2.26 and 2.32 in Ref. [9]), similar to that of Fig. 1(a).

By identifying $1/(2RC)$ with $a$ and $1/(LC)$ with $v^2$ in Eq. (1) [10], the roots of Eq. (7) —the inverse of the time constants—for $a>v$ become:

$$p_{1,2} = -a \pm \sqrt{a^2 - v^2} = -a \pm w_2. \tag{8}$$

A natural extension to the case of $v^2<0$ is to consider the roots as being given by $p_{1,2}=-a\pm w_3$ ($w_3=\sqrt{a^2+|v^2|}$). In this way, we obtain a first noteworthy result: the delay time, as given by

$$t_3 = \frac{1}{a+w_3} = \frac{1}{a+\sqrt{a^2+|v^2|}}, \tag{9}$$

is found to be practically coincident with the one reported as Eq. (5) in Ref. [1] for $l=1$, namely,

$$t_3' = \frac{1}{2a}\left[1 - \exp\left(-\frac{2a}{|v|}\right)\right]. \tag{10}$$

The time as defined by Eq. (10) is a direct consequence of the hypothesis formulated in Ref. [8], according to which the roles of the time variables $t$ (the "true" time) and $\overline{r}$ (the randomized time) are exchanged when passing from classical to tunneling motions as analyzed within the framework of a stochastic model [1–3]. In Fig. 3 (upper part), we compare the delay as given by Eqs. (9) and (10) for $a=0.1$, as a function of $v^2$ in the −0.06–0 range. In order to better understand the dependence on the length $l$ and the frequency $\nu$, it is convenient to rewrite Eq. (9) as

$$t_3 = \frac{\left(\frac{l}{c}\right)}{\left(\frac{al}{c}\right) + \sqrt{\left(\frac{al}{c}\right)^2 + \left(\frac{|v|l}{c}\right)^2}} = \frac{\left(\frac{l}{c}\right)}{\left(\frac{\tilde{a}}{2c}\right) + \sqrt{\left(\frac{\tilde{a}}{2c}\right)^2 + \left(\frac{2|\Delta\nu|}{\nu_0}\right)}}, \tag{11}$$

where $c$ is the light velocity in vacuum, $\nu_0=9.495$ GHz is the cutoff frequency, $\tilde{a}=2al$, and $vl=\tilde{v}$ is the group velocity in waveguide which, for $\nu\sim\nu_0$, is given by

$$\tilde{v} = c\sqrt{(2|\Delta\nu|)/\nu_0} \quad \text{with } \Delta\nu = \nu - \nu_0.$$

Analogously, Eq. (10) becomes

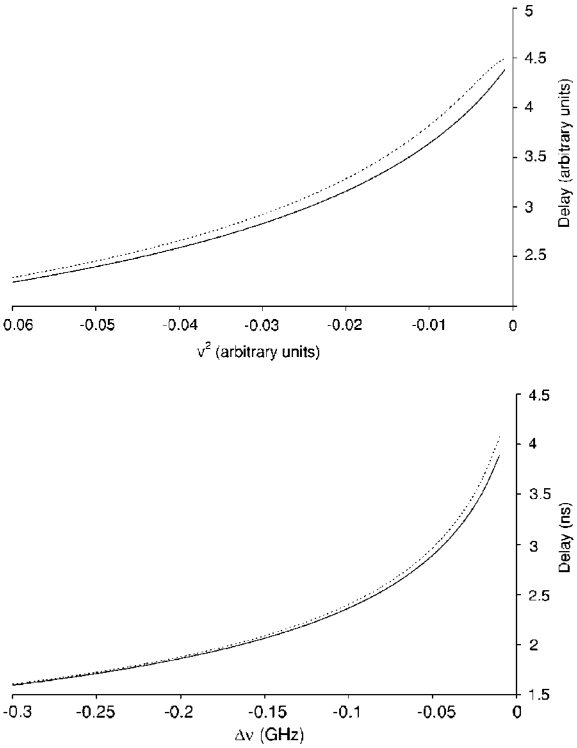


FIG. 3. Delay in arbitrary units, upper part, as given by Eq. (9) (continuous line) and Eq. (10) (dotted line) as a function of $v^2$, for $a=0.1$. In the lower part, the delay in ns as given by Eq. (11) (continuous line) and Eq. (12) (dotted line) as a function of $\Delta\nu=\nu-\nu_0$ in GHz. Parameter values are: $l/c=0.5$ ns and $\tilde{a}/c=0.112$.

$$t_3' = \frac{\left(\frac{l}{c}\right)}{\left(\frac{\tilde{a}}{c}\right)}\left[1-\exp\left(-\frac{\tilde{a}}{|\tilde{v}|}\right)\right]. \qquad (12)$$

The curves calculated with Eqs. (11) and (12) are shown in Fig. 3 (lower part) as a function of $\Delta\nu$, and confirm their very good agreement.

In order to obtain the delay curve for $a>v$, Eq. (9) simply becomes

$$t_2 = \frac{1}{a+w_2} = \frac{1}{a+\sqrt{a^2-v^2}} \qquad (13)$$

and, after the proper substitutions, we have

$$t_2 = \frac{\left(\frac{l}{c}\right)}{\left(\frac{\tilde{a}}{2c}\right)+\sqrt{\left(\frac{\tilde{a}}{2c}\right)^2-\frac{2\Delta\nu}{\nu_0}}} \quad \text{with} \frac{2\Delta\nu}{\nu_0} \leqslant \left(\frac{\tilde{a}}{2c}\right)^2. \qquad (14)$$

A further extension to upper frequencies, that is, for $2\Delta\nu/\nu_0>(\tilde{a}/2c)^2$, can be obtained by considering $w_1$, namely,

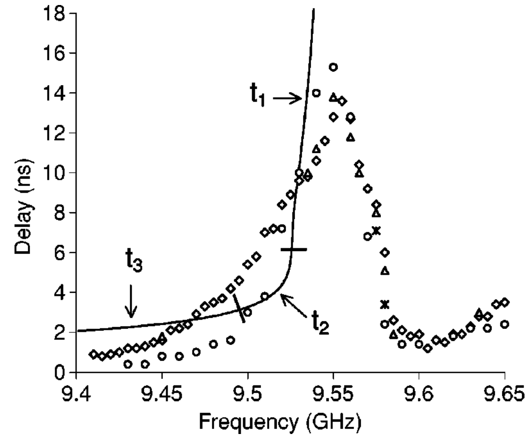


FIG. 4. Delay curve as given by Eqs. (11), (14), and (16) for $l/c=0.5$ ns and $\tilde{a}/c=0.16$, compared with the experimental data in the low frequency region for $l=15$ cm and for different series of measurements (from Ref. [1]).

$$t_1 = \frac{1}{a-w_1} = \frac{1}{a-\sqrt{v^2-a^2}} \quad \text{with } w_1<a, \qquad (15)$$

which, after the aforesaid substitutions, becomes

$$t_1 = \frac{\left(\frac{l}{c}\right)}{\left(\frac{\tilde{a}}{2c}\right)-\sqrt{\frac{2\Delta\nu}{\nu_0}-\left(\frac{\tilde{a}}{2c}\right)^2}}. \qquad (16)$$

When calculated in the appropriate frequency ranges, Eqs. (11), (14), and (16) give rise to a continuous curve, as represented in Fig. 4, where the experimental data are also reported for the case of $l=15$ cm. For the constant $\tilde{a}/c$, the value of 0.160 was selected in order to obtain a reasonable description of the data. However, the agreement remains rather poor mainly because of the presence of a pronounced knee in the theoretical curve, which is not so evident in the experimental data. However, the description of the data is now better than that of Fig. 3 in Ref. [1] where a discontinuity was present in passing from $t_3$ to $t_2$, one which is absent in the present model. The behavior for upper frequencies, described by Eq. (4) in Ref. [1], is not considered here. The present effort is devoted to the interpretation of the experimental data in the tunneling spectral region alone (lower frequency range).

Until now, the role of the two time constants—the inverse of $-p_{1,2}$ as given by Eq. (8)—has been considered separately, obtaining the results shown in Fig. 4. More properly, for a system characterized by two time constants ($\tau_1$ and $\tau_2$), the delay time is given, with a good approximation, by the sum of the two time constants (see Fig. 2.18 in Ref. [9]). In the formulas, we have

$$t_{eff} \simeq \tau_1+\tau_2 = -\frac{1}{p_1}-\frac{1}{p_2} = -\frac{p_1+p_2}{p_1p_2}, \qquad (17)$$

where $-p_{1,2}=a\pm w=a\pm\sqrt{a^2-v^2}$. Thus, $t_{eff}$ is found to be given simply by

$$t_{eff} \simeq \frac{2a}{a^2 - w^2} = \frac{2a}{v^2}, \tag{18}$$

which holds also for $v^2 < 0$, taking for $v^2$ its absolute value. Since Eq. (8) was obtained from Eq. (7) by identifying $1/LC$ with $v^2$, we deduce that $v^2$ must have the dimension of $(\text{time})^{-2}$. In this way, we rightly obtain a time in Eq. (18), where $a$ has the dimension of $(\text{time})^{-1}$ [10]. Analogously to what was done in order to obtain Eq. (11) from Eq. (9) ($\tilde{v} = c\sqrt{2|\Delta\nu|/\nu_0}$ is a true velocity), we have to modify Eq. (18) by including a factor $l^2$ [as already made on Eqs. (11) and (12)], namely,

$$t_{eff} \simeq 2a(l/\tilde{v})^2. \tag{19}$$

The inclusion of $l^2$ can be further justified considering that the delay of a $LC$ cell in an artificial line (that is a series of lumped-constant circuits similar to the one considered here) is given by $\tau_a = \sqrt{LC}$ (see Chap. 10 in Ref. [9]). The delay of a section of length $l$ of a transmission line is given by $\tau_l = l\sqrt{L_0 C_0}$, where $L_0$ and $C_0$ are the inductance and the capacitance per unit length, respectively, and $1/\sqrt{L_0 C_0}$ is the propagation velocity in the line. By identifying $\tau_a$ with $\tau_l$, in order to have a velocity as given by $\tilde{v} = l/\tau$, we must deduce $\tilde{v} = l/\sqrt{LC}$, so that $(l/\tilde{v})^2 = LC$ rightly has dimensions of $(\text{time})^2$.

Equation (19) represents another noteworthy result, since it supplies an expression for the delay time, which is very similar to the one already obtained within the framework of a stochastic model, as well as in that of a path-integral treatment, according to which the delay time is given simply by $at_s^2$, where $t_s$ is the semiclassical time just given by $l/\tilde{v}$ [1].

The continuous line shown in Fig. 5 was obtained from Eq. (19) by taking $a = 0.11$ ns$^{-1}$ and an effective value for the cutoff frequency, which is a little larger than the nominal one [3], namely, $\nu_0 = 9.56$ GHz. Thus, a better description of the experimental data is obtained, also considering that is obtained from a single expression. For the sake of completeness, in Fig. 5 we report also the curve describing the behavior in the upper frequency region (dashed line, taken from Ref. [1]).

We can therefore conclude that this last approach to tunneling time determination gives a very acceptable description of the data, and confirms the prediction already obtained by different approaches to the problem [11,12].

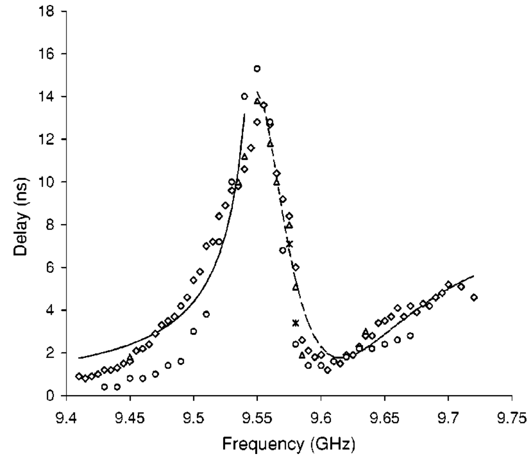


FIG. 5. Delay curve (continuous line) as given by Eq. (19) for $l = 15$ cm, $a = 0.11$ ns$^{-1}$ and $\nu_0 = 9.56$ GHz, compared, always in the low-energy region, with the experimental data. The dashed line, taken from Ref. [1], refers to a previous theoretical model able to well describe the data in the upper region of frequencies.

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[5] S. K. Foong, in *Developments in General Relativity. A Jubilee Volume in Honour of Nathan Rosen*, edited by F. I. Coperstock *et al.* (Institute of Physics, Bristol, 1990), p. 367.
[6] We have to note that the argument of the sinus function should be taken, as usual, $kx - \omega t$, with $k = 2\pi/\lambda$ as the wave number and $\omega = 2\pi v$ the angular frequency. Therefore, the adopted argument $(x - vt)$ (according to the notations of Ref. [2]) has to be considered as normalized to $k$, $v$ being $\omega/k$.
[7] A procedure of this kind was proposed earlier by D. Mugnai, A. Ranfagni, R. Ruggeri, and A. Agresti in Phys. Rev. E **50**, 790 (1994), for the more complicated case of wavelet signals.
[8] The approach of Ref. [1] is indeed different, leading to the expressions for the delay time, Eqs. (4)–(7), which are a direct consequence of the stochastic nature of the motion, both for classically allowed and fordidden cases. The latter, however, still rest on the adoption of the analytical continuation into imaginary time, as applied to the distribution function $g(t,r)$. See D. Mugnai, A. Ranfagni, R. Ruggeri, and A. Agresti, Phys. Rev. E **49**, 1771 (1994).
[9] J. Millman and H. Taub, *Pulse and Digital Circuits* (McGraw-Hill, New York, 1956), Chap. 2.
[10] This is not exact since, as explained below, $1/LC$ has dimensions of $(\text{time})^{-2}$, while $v$ in Eq. (1) is in fact a velocity.
[11] The only difference with respect to the results reported in Ref. [1] (Fig. 4 therein) is an unessential factor of two, which can be neutralized in selecting the value of the parameter $a$.
[12] For a completely different approach to this topic, see also M. Saltzer and J. Ankerhold, Phys. Rev. A **68**, 042108 (2003), where the problem of real time in tunneling processes is developed on the basis of the Feynman path-integral method. The approach of the present paper, which considers a new way to derive the solution of the telegrapher's equation, is totally different.